In a similar manner, the complete solution for the clamped-clamped beam column may be written

$$y = \Sigma \frac{Lc_k(n_k)!}{P\Delta\alpha^{n_k}} (\beta_k \,\mathrm{tru}_2(\alpha x) + \gamma_k \,\mathrm{tru}_3(\alpha x) - \Delta \,\mathrm{tru}_{n_k+2}[\alpha\{x - a_k\}])$$

where

$$\beta_k = \mathrm{tru}_{n_k+2}(\alpha b_k)\cdot \mathrm{tru}_2(\alpha L) - \mathrm{tru}_{n_k+1}(\alpha b_k)\cdot \mathrm{tru}_3(\alpha L)$$
$$\gamma_k = \mathrm{tru}_{n_k+1}(\alpha b_k)\cdot \mathrm{tru}_2(\alpha L) - \mathrm{tru}_{n_k+2}(\alpha b_k)\cdot \mathrm{tru}_1(\alpha L)$$
$$\Delta = \mathrm{tru}_2(\alpha L)\cdot \mathrm{tru}_2(\alpha L) - \mathrm{tru}_1(\alpha L)\cdot \mathrm{tru}_3(\alpha L) = 2 - 2\cos(\alpha L) - \alpha L \sin(\alpha L)$$

References

[1] Urry, S. A., "The use of Macauley's brackets in the analysis of laterally loaded struts and tie-bars," AIAA J. **1**, 462–463 (1963).

[2] Brock, J. E. and Newton, R. E., "A method of pedagogic value in elementary bending theory," Civil Eng. Bull. ASEE **17**, 10–12 (February 1952).

[3] Weissenburger, J. T., "Integration of discontinuous expressions arising in beam theory," AIAA J. **2**, 106–108 (1964).

Natural Frequencies of Meridional Vibration in Thin Conical Shells

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Nomenclature

u_S, u_θ, u_r	=	meridional, tangential, and normal displacement, in.
$\epsilon_S, \epsilon_\theta, \epsilon_{s\theta}$	=	meridional, tangential, and shear strain, in./in.
$\sigma_s, \sigma_\theta, \sigma_{s\theta}$	=	meridional, tangential, and shear stress, psi
G	=	shear modulus, psi
E	=	tensile modulus, psi
ν	=	Poisson's ratio
ρ	=	mass density, #sec²/in.⁴
ω	=	circular frequency, rad/sec
α	=	cone half angle, deg
t	=	cone wall thickness, in.
L^*	=	length of cone along meridian, in.
J_n, Y_n	=	Bessel functions of first and second kind
r_0, r_1	=	cone radii, in.
L	=	length of cone along axis, in.
a	=	$(E/\rho)^{1/2}$ = velocity of sound in the material, in./sec
b	=	$r_0 L^*/r_1 - r_0$, in.
β	=	r_0/r_1
Ω	=	$\omega L^*/a$ = frequency parameter, rad/sec
r_S	=	$r_0 + (r_1 - r_0/L^*)S$, in.; see Fig. 1
A_S	=	$2\pi r_s t$ = cone cross-sectional area, in.²

Introduction

ACOUSTICAL instability, sometimes present in solid propellant rocket motors, may result in the generation of longitudinal vibratory forces of sufficient magnitude to threaten structural integrity of the various motor components. A knowledge of the resonant frequency characteristics of these components in conjunction with predicted acoustic frequencies is therefore of primary importance to the motor designer.

Reasonably accurate estimates can be obtained for resonant frequencies of axial vibration in bars of constant cross section, neglecting lateral inertia effects, using methods readily found in existing literature.[1] In many instances, the motor components to be investigated do not satisfy this condition of constant cross-sectional area. For example, most motor nozzles and some types of interstage structures fall into a category more readily represented by a thin conical shell.

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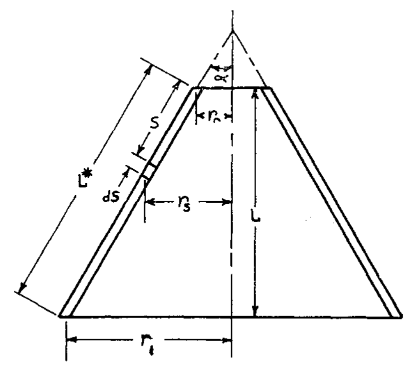

Fig. 1 Conical shell geometry.

The work presented here deals with the prediction of natural frequencies of meridional vibration in conical shells of constant wall thickness. The results can be considered to be an extension of the approximate theory for uniform bars just described since lateral inertia effects have been neglected in the formulation.

Discussion

Consider the thin-walled conical shell of constant wall thickness shown in Fig. 1. The general stress-strain and strain-displacement relations for a thin conical shell are[2]:

$$\sigma_S = \frac{E}{1-\nu^2}(\epsilon_S + \nu\epsilon_\theta) \tag{1}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_S) \tag{2}$$

$$\sigma_{S\theta} = G\epsilon_{S\theta} \tag{3}$$

$$\epsilon_S = \partial u_S/\partial S \tag{4}$$

$$\epsilon_\theta = \frac{1}{r_S}\frac{\partial u_\theta}{\partial\theta} + \frac{u_n \cos\alpha - u_S \sin\alpha}{r_S} \tag{5}$$

$$\epsilon_{S\theta} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial S} + \frac{1}{r_S}\frac{\partial u_S}{\partial\theta} + \frac{u_\theta \sin\alpha}{r_S}\right] \tag{6}$$

The assumption is now made that, during meridional vibration, the cross sections of the cone remain plane, and particles in these cross sections perform only motions in the meridional direction. This results in a system of membrane loading in which

$$\sigma_\theta = \sigma_{S\theta} = \epsilon_{S\theta} = 0$$

Upon substitution of these conditions into Eqs. (1–6) we obtain

$$\sigma_S = E\epsilon_S = E(\partial u_S/\partial S) \tag{7}$$

The meridional force at S is given as

$$F_S = A_S\sigma_S = A_S E(\partial u_S/\partial S) \tag{8}$$

The change in force across the element (ds) is given as

$$dF_S = EdS\,2\pi t\left[r_S\frac{\partial^2 u_S}{\partial S^2} + \frac{r_1 - r_0}{L^*}\frac{\partial u_S}{\partial S}\right] \tag{9}$$

The meridional inertia force of the element at s is given as

$$F_I = -M_S\ddot{u}_S = -2\pi r_S t\rho dS\ddot{u}_S \tag{10}$$

Upon application of D'Alembert's principle, the following differential equation of motion results:

$$\frac{\partial^2 u_S}{\partial S^2} + \frac{1}{b+S}\frac{\partial u_S}{\partial S} = \frac{1}{a^2}\frac{\partial u_S}{\partial t} \tag{11}$$

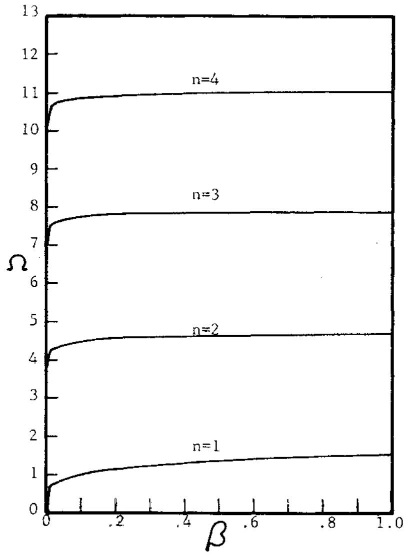

Fig. 2 Fixed-free case.

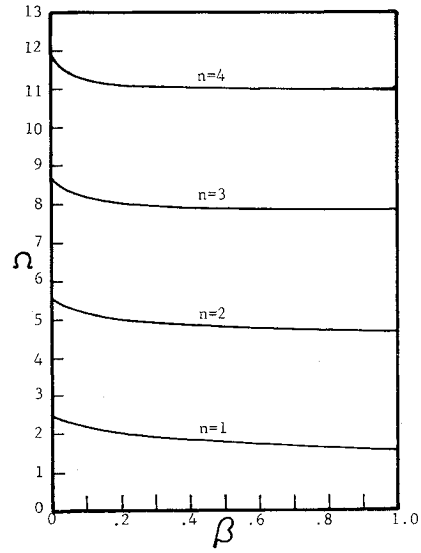

Fig. 3 Free-fixed case.

Next, assume a steady-state solution of the form

$$u_S = Ue^{i\omega t} \tag{12}$$

By substituting Eq. (12) into Eq. (11) we obtain

$$\frac{\partial^2 U}{\partial s^2} + \frac{1}{b+S}\frac{\partial U}{\partial S} + \frac{\omega^2}{a^2}U = 0 \tag{13}$$

Let

$$z = (\omega/a)(b + S)$$

Equation (13) then reduces to

$$z^2 \frac{\partial^2 U}{\partial z^2} + z\frac{\partial U}{\partial z} + z^2 U = 0 \tag{14}$$

Equation (14) is recognized as Bessels' equation of order zero; its solution is given as[3]

$$U = CJ_0(z) + DY_0(z) \tag{15}$$

Upon substitution of Eq. (15) into Eqs. (12) and (4), the solutions for meridional displacements and strains become

$$u_S = \left\{CJ_0\left[\Omega\left(\frac{\beta}{1-\beta} + \frac{S}{L^*}\right)\right] + DY_0\left[\Omega\left(\frac{\beta}{1-\beta} + \frac{S}{L^*}\right)\right]\right\} e^{i\omega t}$$

$$\epsilon_S = -\left\{CJ_1\left[\Omega\left(\frac{\beta}{1-\beta} + \frac{S}{L^*}\right)\right] + DY_1\left[\Omega\left(\frac{\beta}{1-\beta} + \frac{S}{L^*}\right)\right]\right\} e^{i\omega t} \tag{16}$$

The frequency parameter is determined for the following cases of displacement boundary conditions:

Case 1, Fixed-Free

$u_S = 0$ at $S = 0$ $\quad\epsilon_S = 0$ at $S = L^*$

Case 2, Free-Fixed

$\epsilon_S = 0$ at $S = 0$ $\quad u_S = 0$ at $S = L^*$

Case 3, Free-Free

$\epsilon_S = 0$ at $S = 0$ $\quad\epsilon_S = 0$ at $S = L^*$

Case 4, Fixed-Fixed

$u_S = 0$ at $S = 0$ $\quad u_S = 0$ at $S = L^*$

Substitution of these boundary conditions into Eqs. (16) yields the following characteristic equation for each case:

Case 1, Fixed-Free

$$J_0\left(\frac{\Omega\beta}{1-\beta}\right)Y_1\left(\frac{\Omega}{1-\beta}\right) - J_1\left(\frac{\Omega}{1-\beta}\right)Y_0\left(\frac{\Omega\beta}{1-\beta}\right) = 0$$

Case 2, Free-Fixed

$$J_1\left(\frac{\Omega\beta}{1-\beta}\right)Y_0\left(\frac{\Omega}{1-\beta}\right) - J_0\left(\frac{\Omega}{1-\beta}\right)Y_1\left(\frac{\Omega\beta}{1-\beta}\right) = 0$$

Case 3, Free-Free

$$J_1\left(\frac{\Omega\beta}{1-\beta}\right)Y_1\left(\frac{\Omega}{1-\beta}\right) - J_1\left(\frac{\Omega}{1-\beta}\right)Y_1\left(\frac{\Omega\beta}{1-\beta}\right) = 0 \tag{17}$$

Case 4, Fixed-Fixed

$$J_0\left(\frac{\Omega\beta}{1-\beta}\right)Y_0\left(\frac{\Omega}{1-\beta}\right) - J_0\left(\frac{\Omega}{1-\beta}\right)Y_0\left(\frac{\Omega\beta}{1-\beta}\right) = 0$$

The first four eigenvalues of each of these equations plotted as a function of β are shown in Figs. 2–5. Note that, for the free-free case, the first eigenvalue equals zero for all β. Special care must be used in evaluating the eigenvalues for the conditions $\beta = 1$ and $\beta = 0$.

$\beta = 1$: For this condition, the cone transforms into a cylinder as the arguments of the Bessel functions in the characteristic equations approach infinity. The Bessel functions can be evaluated from the following relations[3]:

$$\lim_{x\to\infty} J_n(x) = \frac{\cos[x - (\pi/4) - (n\pi/2)]}{(\pi x/2)^{1/2}}$$

$$\lim_{x\to\infty} Y_n(x) = \frac{\sin[x - (\pi/4) - (n\pi/2)]}{(\pi x/2)^{1/2}} \tag{18}$$

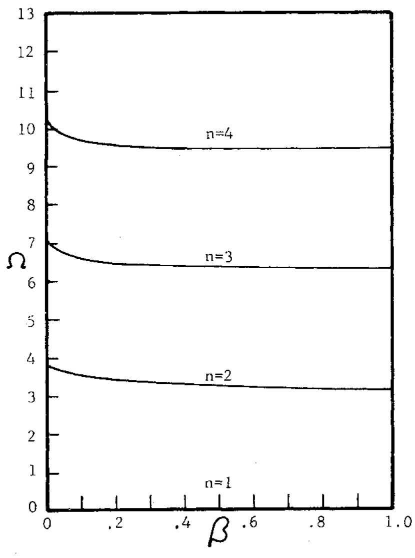

Fig. 4 Free-free case.

The values of Ω, as shown in Figs. 2–5 for $\beta = 1$, correspond exactly to those for circular bars with the same boundary conditions.

$\beta = 0$: The solutions for the free-fixed and free-free cases are determined as follows: Examination of Eq. (16) reveals that, in both cases, for a solution to exist, the constant D must equal zero; therefore, Eq. (17) can be rewritten as

$$(u_S)_{\beta=0} = CJ_0(\omega S/a)e^{i\omega t}$$
$$(\epsilon_S)_{\beta=0} = -CJ_1(\omega S/a)e^{i\omega t} \quad (19)$$

Substitution of the required boundary conditions yields, for the free-fixed case, $J_0(\Omega) = 0$ and, for the free-free case, $J_1(\Omega) = 0$. The limiting values of the frequency parameter Ω for the fixed-free and fixed-fixed cases are found from Eqs. (17): $J_1(\Omega) = 0$ for the fixed-free case and $J_0(\Omega) = 0$ for the fixed-fixed case.

Summary

The first four modes of the meridional vibrations in a conical shell of constant wall thickness have been determined for various cases of displacement boundary conditions.

The solution as presented is approximate in that the effects of lateral inertia are not included. Abramson, Plass, and Ripperger[4] compare the exact solutions for axial wave propagation in rods of circular cross section with an approximate solution that does not include the effect of radial inertia. They conclude that the approximate solution is reasonably accurate for pulses of very long duration, that is, where duration of the pulse is long compared with the time required for a wave front traveling at velocity a to move a distance equal to the diameter of the bar. If we apply an analogous line of reasoning to the cone problem, it would appear that the solution as presented is reasonably accurate if the ratio meridian length (pulse length) to wall thickness (bar diameter) is large. This condition should not be found overly restrictive in most engineering problems of practical interest.

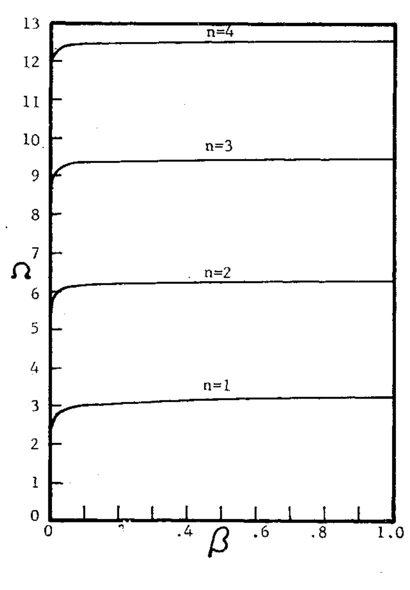

Fig. 5 Fixed-fixed case.

References

[1] Den Hartog, J. P., *Mechanical Vibrations* (McGraw-Hill Book Co., Inc., New York, 1956), p. 481.

[2] Timoshenko, S., *Theory of Plates and Shells* (McGraw-Hill Book Co., Inc., New York, 1940), Chap. X.

[3] Pipes, L. A., *Applied Mathematics for Engineers and Physicist* (McGraw-Hill Book Co., Inc., New York, 1946), Chap. XIII.

[4] Abramson, H. N., Plass, J. H., and Ripperger, E. A., "Stress wave propagation in waves and beams," *Advances in Applied Mechanics*, edited by H. L. Dryden and T. Von Karman (Academic Press Inc., New York, 1958), Vol. 4, pp. 113–138.

Mismatch Stresses in Pressure Vessels

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In the manufacture of pressure vessels it is often necessary to join segments together at circumferential joints by welds or other means. Such connections may be made in lightweight flight tanks by lapping the segments and spot welding (Fig. 1a) or by butt welding the adjacent segments (Fig. 1b). The purpose of this note is to present an approximate analysis for the stresses arising at such a joint because of the mismatch or nonconcurrence of the middle surfaces of the adjoining segments which might occur. The method used is applicable to any nonshallow shell of revolution in which the meridional tangents of the two segments are parallel to each other at the junction.

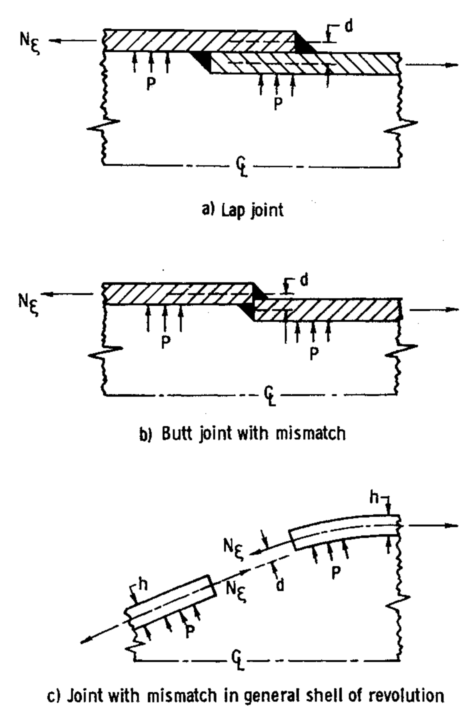

Fig. 1 Mismatch joints and loading.

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